Warsaw University of Technology

Faculty of Power and Aeronautical Engineering
WARSAW UNIVERSITY OF TECHNOLOGY

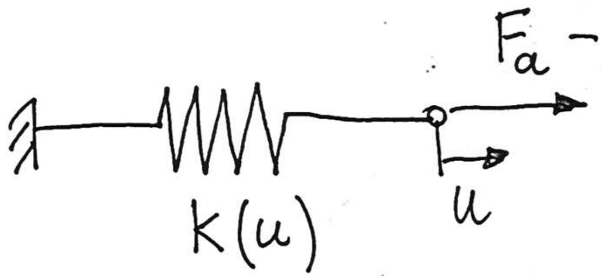
Institute of Aeronautics and Applied Mechanics

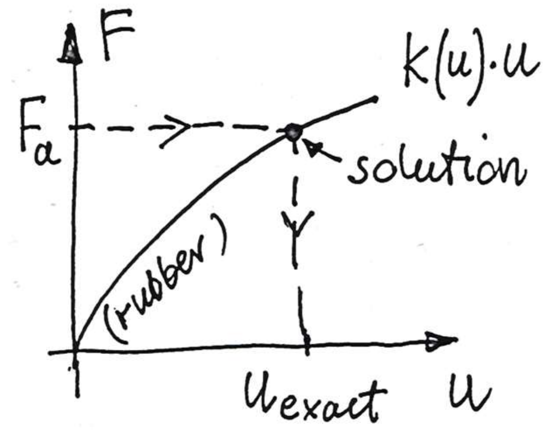
Finite element method 2 (FEM 2)

Iterative solution

11.2021

ITERATIVE SOLUTION OF A NONLINEAR EQUATION





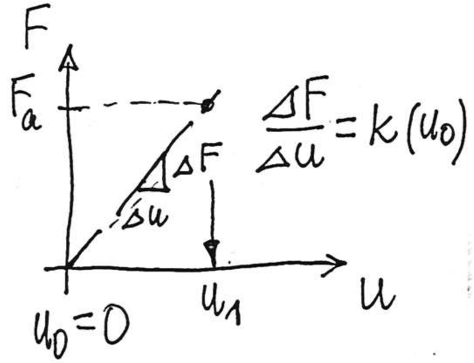
nonlinear stiffness: $k(u) = k_o - c \cdot u$

$k_o \left[\frac{N}{m}\right]$, $c = \left[\frac{N}{m^2}\right]$

1°) Direct approach :

- initial solution : $u_0 = 0 \Rightarrow k(u_0) = k_0 - c \cdot 0 = k_0$

- 1st iteration : $u_1 = \dfrac{F_a}{k(u_0)}$



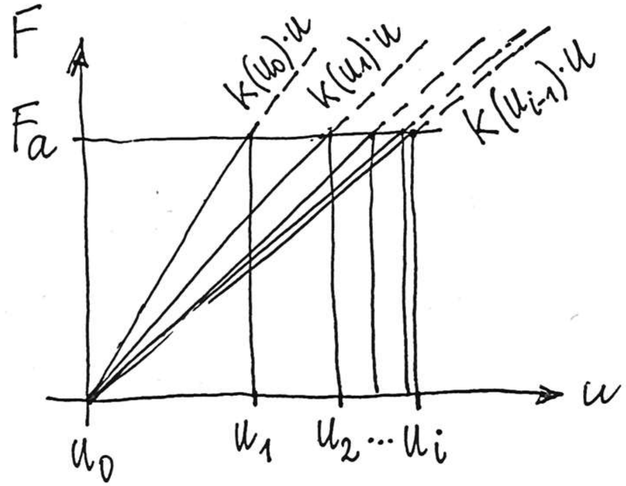
- iteration „i" : $u_i = \dfrac{F_a}{k(u_{i-1})}$

increment of displacement : $\Delta u_i = u_i - u_{i-1}$

convergence criterion :

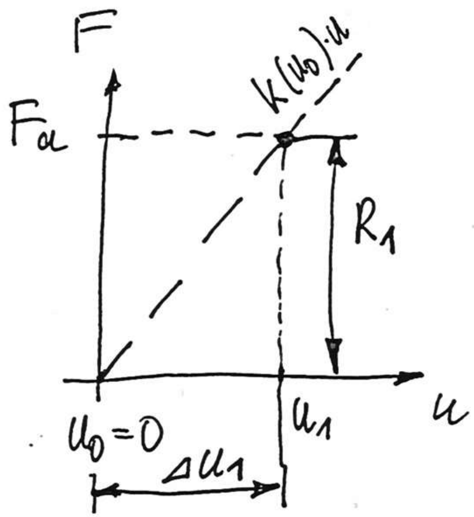
$$\frac{\Delta u_i}{u_i} \leq \varepsilon$$

↑ tolerance



2°) Direct incremental approach :

- initial solution : $u_0 = 0 \quad \Rightarrow \quad k(u_0) = k_0 - c \cdot 0 = k_0$



- 1st iteration :

  residual force : $R_1 = F_a - k(u_0) \cdot u_0 = F_a$

  increment of displacement : $\Delta u_1 = \dfrac{R_1}{k(u_0)}$

  displacement : $u_1 = \Delta u_1 + u_0 = \Delta u_1$

- iteration „i" : $R_i = F_a - k(u_{i-1}) \cdot u_{i-1}$

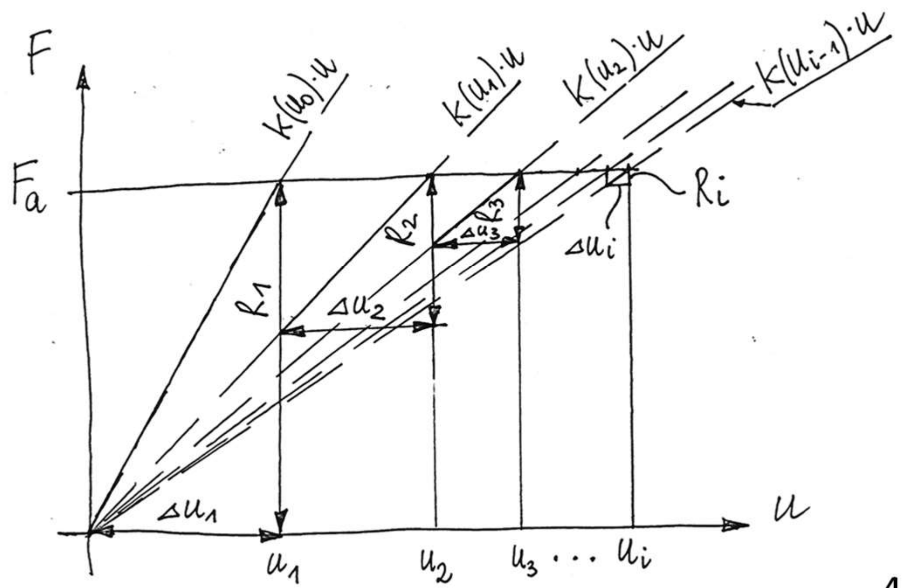
$$\Delta u_i = \frac{R_i}{k(u_{i-1})}$$

$$u_i = \Delta u_i + u_{i-1}$$

convergence criteria :

$$\frac{\Delta u_i}{u_i} \leq \varepsilon \quad \text{and} \quad \frac{R_i}{F_a} \leq \delta$$

tolerances



3°) Newton - Raphson method:

tangent stiffness : $k_T = \frac{dF}{du} = \frac{d(k(u)\cdot u)}{du}$

$= \frac{dk(u)}{du}\cdot u + \frac{du}{du}\cdot k(u) = -c\cdot u + k_0 - c\cdot u = k_0 - 2c\cdot u$

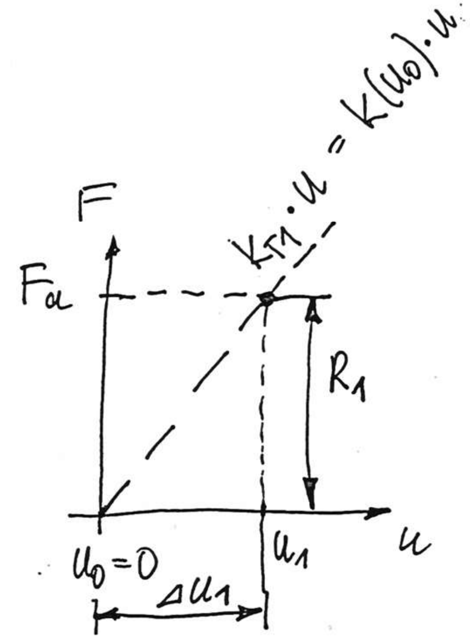
- initial solution : $u_0 = 0 \quad \Rightarrow \quad k(u_0) = k_0 - c\cdot 0 = k_0$

- 1st iteration : $k_{T1} = \left.\frac{dF}{du}\right|_{u_0} = k_0 - 2c\cdot u_0 = k_0$

  residual force : $R_1 = F_a - k(u_0)\cdot u_0 = F_a$

  increment of displacement : $\Delta u_1 = \frac{R_1}{k_{T1}}$

  displacement : $u_1 = \Delta u_1 + u_0 = \Delta u_1$



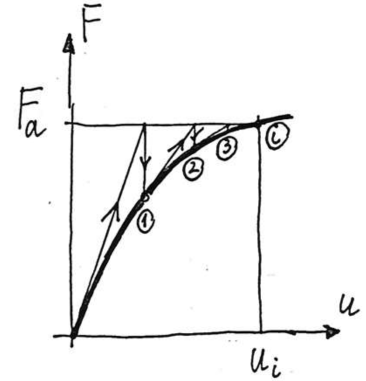
– iteration „ i " :

$$K_{Ti} = \left.\frac{dF}{du}\right|_{u_{i-1}} = k_0 - 2c \cdot u_{i-1}$$

$$R_i = F_a - k(u_{i-1}) \cdot u_{i-1}$$

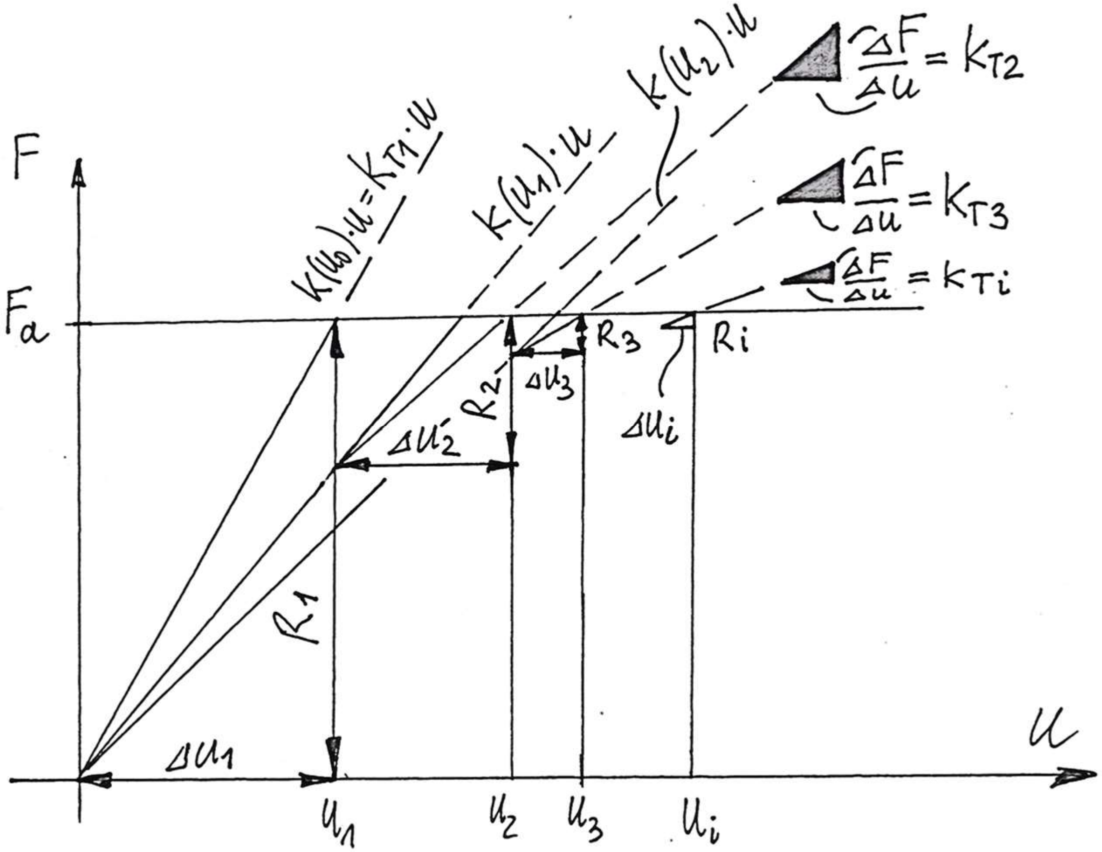
$$\Delta u_i = \frac{R_i}{k_{Ti}}$$

$$u_i = \Delta u_i + u_{i-1}$$



convergence criteria :

$$\underbrace{\frac{\Delta u_i}{u_i} \leq \varepsilon \quad \text{and} \quad \frac{R_i}{F_a} \leq \delta}_{\text{tolerances}}$$



FULL NEWTON - RAPHSON METHOD FOR A SET OF NONLINEAR EQUATIONS INCLUDING BOUNDARY CONDITIONS :

$$\underset{N\times N}{\left[K(q)\right]} \cdot \underset{N\times 1}{\left\{q\right\}} = \underset{N\times 1}{\left\{F\right\}}$$

where : $N = NDOF - NOF$

initial solution :

$$\underset{N\times 1}{\left\{q\right\}_0} = \underset{N\times 1}{\left\{0\right\}} \Rightarrow \underset{N\times N}{\left[K(q)\right]_0}$$

tangent matrix at iteration „i":

$$\underset{N\times N}{[K_T]}_i = \frac{\partial \underset{N\times 1}{\{F\}}}{\partial \underset{1\times N}{\lfloor q \rfloor}_{i-1}} =$$

$$= \left[\frac{\partial \underset{N\times 1}{\{F\}}}{\partial q_1}, \frac{\partial \underset{N\times 1}{\{F\}}}{\partial q_2}, \dots, \frac{\partial \underset{N\times 1}{\{F\}}}{\partial q_N} \right]_{i-1}$$

where: $$\frac{\partial \underset{N\times 1}{\{F\}}}{\partial q_j} = \frac{\partial\left(\underset{N\times N}{[K(q)]} \cdot \underset{N\times 1}{\{q\}}\right)}{\partial q_j} =$$

$$= \frac{\partial \underset{N\times N}{[K(q)]}}{\partial q_j} \cdot \underset{N\times 1}{\{q\}} + \underbrace{\underset{N\times N}{[K(q)]} \cdot \frac{\partial \underset{N\times 1}{\{q\}}}{\partial q_j}}_{\text{column } j \text{ of matrix } \underset{N\times N}{[K(q)]}}$$

$j = 1, \dots, N$

residual vector at iteration „i":

$$\underset{N\times 1}{\{R\}_i} = \underset{N\times 1}{\{F\}} - \underset{N\times N}{[K(q)]_{i-1}} \cdot \underset{N\times 1}{\{q\}_{i-1}}$$

increment of the global vector of nodal parameters:

$$\underset{N\times 1}{\{\Delta q\}_i} = \underset{N\times N}{[K_T]_i^{-1}} \cdot \underset{N\times 1}{\{R\}_i}$$

the global vector of nodal parameters:

$$\underset{N\times 1}{\{q\}_i} = \underset{N\times 1}{\{q\}_{i-1}} + \underset{N\times 1}{\{\Delta q\}_i}$$

convergence criteria :

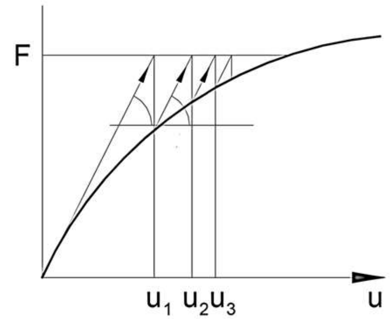
displacement criterion : $$\frac{\| \{\Delta q\}_i \|_2}{\| \{q\}_i \|_2} \leq \varepsilon$$

and

force criterion : $$\frac{\| \{R\}_i \|_2}{\| \{F\} \|_2} \leq \delta$$

MODIFIED NEWTON - RAPHSON METHOD :

$$\underset{N\times N}{[K_T]_i} = \underset{N\times N}{[K_T]_1} = \frac{\partial \underset{N\times 1}{\{F\}}}{\partial \underset{1\times N}{\lfloor q \rfloor_0}}$$



EXAMPLE .

$F_a = 0.2\,N$

$k(u) = k_0 - c \cdot u$ u $k_0 = 1\frac{N}{m}$, $c = 1\frac{N}{m^2}$

Direct incremental approach

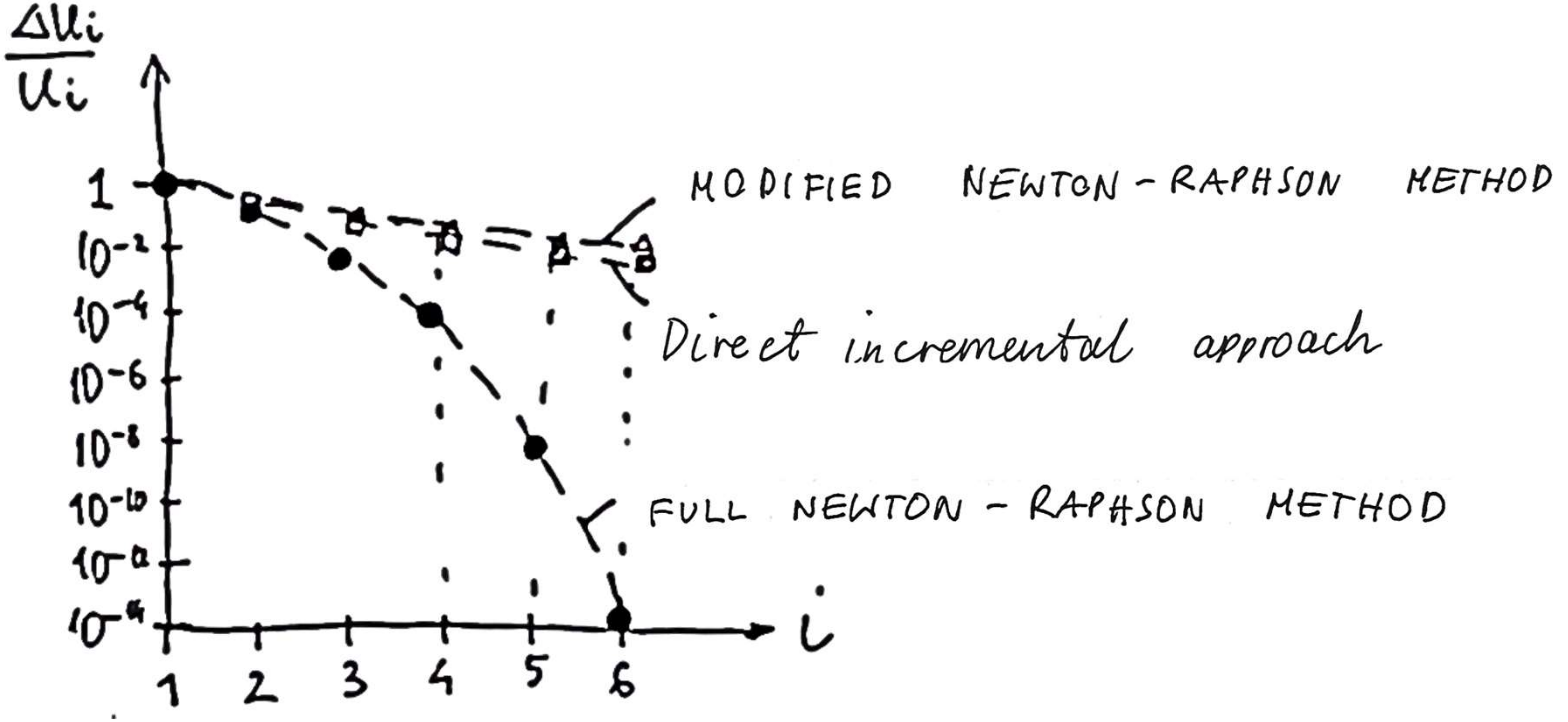
i	u_{i-1}	$k_{i-1} = 1 - u_{i-1}$	$R_i = F - k_{i-1}u_{i-1}$	$\Delta u_i = R_i / k_{i-1}$	$u_i = u_{i-1} + \Delta u_i$	$\frac{\Delta u_i}{u_i}$	$\frac{R_i}{F}$
1	0	1	0.2	0.2	0.2	1	1
2	0.2	0.8	0.04	0.05	0.25	0.2	0.2
3	0.25	0.75	0.0125	0.0167	0.2667	0.063	0.063
4	0.2667	0.733	0.0044	0.006	0.2727	0.022	0.022
5	0.2727	0.7273	0.0017	0.0023	0.2750	0.008	0.0085

FULL NEWTON - RAPHSON METHOD

i	u_{i-1}	$k_{i-1} = 1 - u_{i-1}$	$R_i = F - k_{i-1}u_{i-1}$	$k_T = 1 - 2u_{i-1}$	$\Delta u_i = R_i / k_T$	$u_i = u_{i-1} + \Delta u_i$	$\frac{\Delta u_i}{u_i}$	$\frac{R_i}{F}$
1	0	1	0.2	1	0.2	0.2	1	1
2	0.2	0.8	0.04	0.6	0.0667	0.2667	0.250	0.2
3	0.2667	0.7333	0.0044	0.466	0.0095	0.2762	0.048	0.034
4	0.2762	0.7238	0.0001	0.448	0.0002	0.2764	0.001	0.0005

MODIFIED NEWTON - RAPHSON METHOD

i	u_{i-1}	$k_{i-1} = 1 - u_{i-1}$	$R_i = F - k_{i-1}u_{i-1}$	$\Delta u_i = R_i / k_0$	$u_i = u_{i-1} + \Delta u_i$	$\frac{\Delta u_i}{u_i}$	$\frac{R_i}{F}$
1	0	1	0.2	0.2	0.2	1	1
2	0.2	0.8	0.04	0.04	0.24	0.167	0.2
3	0.24	0.76	0.0176	0.0176	0.2576	0.068	0.088
4	0.2576	0.7424	0.0087	0.00876	0.2664	0.033	0.044
5	0.2664	0.7336	0.0046	0.0046	0.2710	0.017	0.023
6	0.2710	0.729	0.0024	0.0024	0.2734	0.009	0.012



Analytical solution

$$k(u)u = F, \quad u^2 - u + F = 0$$

$$u_1 = \frac{1-\sqrt{1-4F}}{2} = 0.2734,$$

$$u_2 = \frac{1+\sqrt{1-4F}}{2} = 0.7236.$$